Taking Schrödinger Seriously
Module 2

The Wave Function and Hilbert Space

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Introduction

Module 1: What is the Schrödinger Equation? broke down the Schrödinger equation symbol by symbol. In the first module, we learned that the wave function is a complex-valued function that describes the physical state of a quantum system, and that it has a magnitude and a phase, similar to an arrow on the complex plane.

You might be wondering: where does this wave function actually *live*? The wave function of a single particle isn't floating around in the room with you, meaning that it does not exist in ordinary three-dimensional space. The wave function exists in a different kind of space altogether, called *Hilbert space*.

Hilbert space is more than just mathematics. It's the key to understanding why quantum mechanics behaves the way it does. Specifically, Hilbert space introduces the concept of orthogonality—the idea that some quantum states absolutely do not overlap. Orthogonality is going to be crucial for understanding measurement and branching later in the course.

By the end of this module, you will be able to:

- Explain what it means for *wave functions* to be *vectors*
- Describe *Hilbert space* and its key properties
- Define the *inner product* and explain what it measures
- Explain *orthogonality* and why it matters for quantum mechanics
- Distinguish between *finite and infinite* dimensional Hilbert spaces

A Quick Overview of Vectors

Before getting into Hilbert space, we first need to understand vectors. If you've seen vectors before, this will be a quick refresher; if you haven't, this overview covers all that you need to know.

A vector is an arrow with two properties: length, or how long it is, and direction, the course along which it points.

When dealing in two dimensions, a vector is described by its horizontal and vertical positions. Therefore, the vector $(3,\ 2)$ "goes 3 units to the right and 2 units up." Note that because the horizontal value is positive, the vector goes to the right; if it were a negative value, the vector would instead go to the left. So, the vector $(1,\ 0)$ only points to the right, and the vector $(0,\ 1)$ only points up.

When dealing in three dimensions, we add a third number to the vector, the forwards-backwards position. The vector $(3,\ 2,\ 1)$ "goes 3 units to the right, 2 units up, and 1 unit forward."

The addition operation can be applied to vectors, through vector addition. Think of it as placing the arrows of two vectors tip-to-tail, and then drawing a new arrow from the start of the first to the end of the second. Mathematically, it looks like this:

$$(3,\ 2)\ +\ (1,\ 4)\ =\ (4,\ 6)$$

The multiplication operation can also be applied to vectors, through vector scaling. For example, if you multiply a vector by a factor of 2, the vector maintains its orientation but its arrow is now twice as long. If you were to multiply the vector by a factor of $-\ 1$, the orientation of the vector arrows would be flipped.

We now come to a key concept: *the dot product*.

The dot product takes two vectors, measures the extent to which the vectors point in the same direction, and returns a single value.

For two-dimensional vectors, the formula is simple:

$$(a,\ b)\cdot(c,\ d)\ =\ ac\ +\ bd$$

For example:

$$(3,\ 2)\cdot(1,\ 4)\ =\ 3\times 1\ +\ 2\times 4\ =\ 3\ +\ 8\ =\ 11.$$

If two vectors point in the same direction, the dot product is large and positive, and if they point in opposite directions, it's large and negative. If the two vectors are perpendicular—at right angles to each other—the dot product is exactly zero.

The instance of perpendicular vectors is a case of orthogonality. Orthogonality means "completely independent, zero overlap." So, when two vectors have a dot product of zero, the vectors are orthogonal.

For example:

$(1,\ 0) \cdot (0,\ 1)\ =\ 1 \times 0\ +\ 0 \times 1\ =\ 0.$

These vectors are orthogonal—one points right, the other points up, and thus have nothing in common. Orthogonality, independence, and the absence of overlap are crucial concepts for understanding quantum mechanics.

Wave Functions Are Vectors

In the above section, we conceptualized vectors as arrows with a length and a direction. Then, we saw that vectors can be added (tip-to-tail), scaled (stretched or shrunk), and compared using the dot product.

As you may have noticed, we never needed the arrows.

The addition, scaling, and dot products defined the vectors in terms of the components: $(3,\ 2)$, $(1,\ 4)$, and so on. The mathematics is strictly concerned with the operations.

The takeaway is that a vector is really just anything that you can add and scale. Arrows are one example of vectors, but there are many more.

Consider polynomials. When dealing with polynomials. you can add $x^2\ +\ 2x$ and $3x\ +\ 1$ to get $x^2\ +\ 5x\ +\ 1$, and you can scale x^2 by 3 to get $3x^2$. Polynomials can be added and scaled, so, polynomials are also vectors.

Technically, a vector space consists of a collection of objects that can be added and scaled. The collection of all arrows in two dimensions is a vector space, as is the collection of all polynomials. Any collection where addition and scaling are possible is a vector space.

What about wave functions?

First, we know that wave functions can be added. If you add $\Psi_1(x)$ and $\Psi_2(x)$, you get the valid wave function $\Psi_1(x) + \Psi_2(x)$.

Second, wave functions can be scaled. If you multiply $\Psi(x)$ by a complex number c, you get the valid wave function $c\Psi(x)$.

So, wave functions can be added and scaled, meaning that wave functions are vectors, and the collection of all wave functions is a vector space.

This means that when physicists say "the wave function is a vector," they mean it in the precise mathematical sense; they mean that wave functions live in a vector space.

The vector space where wave functions live is a Hilbert space, which is properly introduced in the next section.

Hilbert Space

Not every vector space is a Hilbert space. A vector space becomes a Hilbert space with the addition of extra structure: an inner product.

Remember that the dot product covered above took two vectors and returned a number representative of their overlap. The inner product is the same idea, but generalized to work in any vector space—including spaces of functions.

So, a Hilbert space is a vector space equipped with an inner product.

That's almost the full definition, but there's one more technical requirement: the space must be "complete," meaning that it does not have any holes or gaps. If you take a sequence of vectors that are approaching each other and getting closer and closer together, their limit must also be in the space in order to be considered “complete”. The technical condition of completeness ensures that the mathematics function properly, but it’s a bit out of our depth. For our purposes, you can assume that the spaces in quantum mechanics satisfy this requirement.

Therefore, the full definition is Hilbert space = vector space + inner product + completeness.

Why does understanding the Hilbert space matter for quantum mechanics?

In Module 1, we said that the wave function describes the state of a quantum system. Now, we can be more precise and say that the state of a quantum system is a vector in Hilbert space.

Importantly, this statement has consequences. The presence of an inner product in Hilbert space allows us to quantify the overlap between two quantum states. This overlap has physical meaning, as it determines the probability of finding one state when measuring another.

Since Hilbert space is a vector space, we can add quantum states together, resulting in superposition. If $\Psi_1(x)$ and $\Psi_2(x)$ are valid states, then so is the superposition $\Psi_1(x) + \Psi_2(x)$.

The rules of Hilbert space are the rules of quantum mechanics. Superposition, interference, and measurement probabilities are determined by the structure of Hilbert space.

In the following section, we'll look at the inner product in detail and see exactly how it connects to physical predictions.

The Inner Product

The dot product measures overlap between arrows. Two arrows pointing the same way have a large dot product. Two arrows at right angles have a dot product of zero.

The inner product generalizes this idea. In Hilbert space, it takes two wave functions and returns a number that measures their overlap.

However, the fact that wave functions are complex-valued introduces a complication that the inner product must account for.

The inner product function for the two wave functions $\Psi(x)$ and $\Phi(x)$ is:

$$\langle \Phi \mid \Psi \rangle = \int_{-\infty}^{+\infty} \Phi^*(x)\Psi(x)dx$$

Don't fret! Let us unpack this.

The integral sign means that we're adding up contributions from every point x. At each point, we multiply $\Psi(x)$ by $\Phi^*(x)$, which is the complex conjugate of $\Phi(x)$. The complex conjugate flips the sign of the imaginary part, b, such that if $\Phi = a + bi$, then $\Phi^* = a - bi$.

Why the complex conjugate? The inner product of a wave function with itself could be negative or even complex without it. Therefore, the conjugate ensures that $\langle \Phi \mid \Phi \rangle$ is always a non-negative number—which is good, because the quantity represents total probability, and you cannot have a negative probability.

The notation $\langle \Phi \mid \Psi \rangle$ is called bra-ket notation, introduced by physicist Paul Dirac. The left side, $\langle \Phi |$, is called a "bra," and the right side, $| \Psi \rangle$, is called a "ket." Together they form a "bracket"—quite the pun for a physicist!

The bra-ket notation is abundant in quantum mechanics because it's both compact and powerful. For now, you can simply read $\langle \Phi \mid \Psi \rangle$ as "the overlap between Φ and Ψ."

What does this overlap tell us?

It tells us that if $\langle \Phi \mid \Psi \rangle$ is large, then the two wave functions are similar, or "pointing in similar directions" in Hilbert space. On the other hand, if $\langle \Phi \mid \Psi \rangle$ is zero, the wave functions have nothing in common; they're orthogonal.

One case is especially important: the inner product of a wave function with itself.

$$\langle \Psi \mid \Psi \rangle = \int_{-\infty}^{+\infty} |\Psi(x)|^2 dx$$

This is the integral of $|\Psi(x)|^2$ over all of space—which we know from Module 1 is the total probability of finding the particle somewhere. For a properly normalized wave function, this equals 1.

So, the inner product $\langle \Psi \mid \Psi \rangle = 1$ is just another way of saying that the "total probability equals 100%."

Orthogonality

Two arrows are orthogonal if they're perpendicular—at right angles to each other, and, as a result, their dot product is zero.

Two wave functions are orthogonal if their inner product is zero: $\langle \Psi \mid \Psi \rangle = 0$.

In both cases, orthogonality means that there is zero overlap between the vectors.

For arrows, the meaning is geometric: an arrow pointing east has nothing in common with an arrow pointing north, they share no component, and, if you project one onto the other, you get zero.

For wave functions, the meaning is physical: orthogonal states are mutually exclusive.

What does this mean?

Suppose that Ψ and Φ are orthogonal wave functions. If a quantum system is in state Ψ, the probability of finding it in state Φ is zero. The states have nothing in common, and a system in one state is definitely not in the other.

Let's consider a concrete example: electron spin.

An electron can spin "up" or "down" along any axis. Let's call the up state $|\uparrow\rangle$ and the down state $|\downarrow\rangle$. These states are orthogonal:

$$\langle \uparrow \mid \downarrow \rangle = 0$$

If you measure an electron and find it spin-up, the probability that it is spin-down is zero. These are mutually exclusive outcomes because the electron is measured as being in one state or the other, but never both.

How does this make sense when, as we covered above, quantum systems can be in superpositions? It's possible that the electron is in a state such as this:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

The electron's state is a superposition of up and down, but also a state where $|\uparrow\rangle$ and $|\downarrow\rangle$ remain orthogonal. The superposition doesn't mix them—it simply indicates that the system has components pointing in both directions in Hilbert space.

When you measure an electron, you always get a single result—either up or down, never a mix of the two. The measurement outcomes are distinct because the two states are orthogonal—they don't overlap.

So, orthogonal directions in Hilbert space correspond to mutually exclusive physical possibilities.

Spin up and spin down. Particle here and particle there. A detector that clicked and a detector that didn't click. These pairs of outcomes correspond to orthogonal states.

Dimensions

In ordinary three-dimensional space, you need three values to specify a coordinate: left-right, forward-backward, up-down.

What about in Hilbert space? How many numbers do you need to specify a quantum state?

In Hilbert space, it depends on the system, and the answer can be infinite.

First, we need the concept of a basis. A basis is a set of orthogonal states that spans the entire space—meaning any state can be written as a combination of them. In 3D space, the three directions "right," "up," and "forward" form a basis: any position can be described as some amount of each. The number of basis states tells you the dimension of the space.

The case of electron spin was straightforward. The states $|\uparrow\rangle$ and $|\downarrow\rangle$ form a basis—they're orthogonal, and any spin state can be written as a combination of them:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

Two complex numbers completely determine the state, meaning that this Hilbert space is two-dimensional—the quantum version of a plane.

Now, let's consider a harder case: the position of a particle. A particle can be situated anywhere on a line, and its wave function, $\Psi(x)$, assigns a complex number to every point x. How many numbers does it require to determine the position? Infinitely many—one for each point.

In this case, this Hilbert space is infinite-dimensional.

The basis states for position are wave functions localized at a single point: $|x\rangle$. As every point on the line has a basis state, there are infinitely many, all of which are orthogonal to each other.

The wave function can be written as a combination of these position basis states:

$$|\Psi\rangle = \int |\Psi(x)|x\rangle \, dx$$

Here, the integral replaces the sum because we're adding up an infinity of contributions.

The spin case and the particle space follow the same pattern: pick a set of orthogonal basis states; any state is a combination of them, and the number of basis states is the dimension. For the spin, this results in two dimensions, and for the particle, this results in infinite dimensions.

Infinite-dimensional spaces are harder to visualize, but the concepts of addition, scaling, inner products, and orthogonality carry the same physical meaning as they do for finite-dimensional spaces.

Why This All Matters

So far, we've covered a lot of abstraction: vectors, Hilbert space, inner products, orthogonality, and dimensions.

Why does any of this matter for understanding quantum mechanics? As stated above, the structure of Hilbert space *is* the structure of quantum mechanics.

Superposition is vector addition. Saying that a particle is "in a superposition of here and there" means that its state vector has components in multiple directions—some pointing toward $|here\rangle$, some toward $|there\rangle$.

Interference arises when parts of a wave function overlap in space. At each point, the complex amplitudes add. If they line up, they reinforce; if they point opposite ways, they cancel. The resulting amplitude distribution—where the wave function is large or small—depends on these phase relationships.

Measurement probabilities are derived from the inner product. The probability of finding a system in state $|\Phi\rangle$ when it's in state $|\Psi\rangle$ is $|\langle\Phi \mid \Psi\rangle|^2$, the squared magnitude of their overlap.

Distinct outcomes correspond to orthogonal states. So, when you measure spin and get "up," that doesn't mean that you get a blend of up and down. The orthogonality of $|\uparrow\rangle$ and $|\downarrow\rangle$ prohibits the outcomes from overlapping.

Contrast this with states that aren't orthogonal. A particle with spin pointing sideways is in the state:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

This state has nonzero overlap with both $|\uparrow\rangle$ and $|\downarrow\rangle$. If you measure this particle's spin (up vs down), both outcomes have amplitude. Orthogonality is what makes outcomes mutually exclusive; without it, multiple outcomes remain possible.

Time evolution is motion through Hilbert space. The Schrödinger equation tells you how the state vector moves—it rotates and flows according to precise rules determined by the system's energy.

In Module 3, we'll look at the engine that drives this evolution: the Hamiltonian.

Conclusion

In conclusion, the wave function lives in Hilbert space—a vector space equipped with an inner product.

Every core feature of quantum mechanics maps onto Hilbert space structure:

- Vectors → quantum states
- Vector addition → superposition
- Inner product → overlap and probability
- Orthogonality → mutually exclusive outcomes
- Dimension → how many independent states exist

Hilbert space can be finite-dimensional (electron spin and its two basis states) or infinite-dimensional (particle position and its infinite basis states); the mathematics works the same either way.

The most important concept to remember from this section is that orthogonal states have zero overlap—they represent possibilities that cannot coexist in a single outcome. This concept is going to be essential when we examine measurement and branching.

Next, we turn to the Hamiltonian—the operator that determines how quantum states evolve through Hilbert space.